6A Contest

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A. Bermutation

In this problem we have a permutation p, the number b and we are allowed to make the next operation: take i such that $i \geqslant 1; i + 2b - 1 \leqslant n$, and swap p_i with p_{i+b}, p_{i+1} with p_{i+1+b}, $\ldots$, p_{i+b-1} with p_{i+2b-1}.

We are asked to find the number of permutation p in the lexicographically ordered list of all permutations which can be obtained by the described operations, modulo 120 586 241.

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Another observation is that our operation changes the number of inversions in each slice of our permutation (elements with indices $i, i + b, i + 2b, \ldots$) modulo b exactly by 1, so we can determine the oddity of inversions numbers in each slice if we know oddity of inversions number in one slice.

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These two observations give us an upper bound to the number of achievable permutations: we can achieve no more than $2^{-(b-1)} \prod_{i=0}^{b-1} (\lceil \frac{n+i}{b} \rceil)!$ different permutations.

A. Bermutation

We can prove that this bound is achievable: if $n \leqslant 2b$, we can put any element from positions $0, b, \ldots,$ on a first place and proceed to $n-1$. When we have $n = 2b-1$, there are exactly $b-1$ remainders modulo b with only one from two potentially achievable permutations. So, we can get exactly $\prod_{i=0}^{b-1} (\lceil \frac{n+i}{b} \rceil)!$ (almost permutation at each slice modulo b) divided by 2^{b-1} (the number of permutations that we can't get).

A. Bermutation

Now using the obtained formula we can calculate the number of given permutation. We can fix the position j of the first different element between our permutation and some lesser achievable permutation, count the number l of smaller elements that we can put at the position j, and add the $l \cdot 2^{-(b-1)} \prod_{i=0}^{b-1} (\lceil \frac{n-j-1+i}{b} \rceil)!$ to the answer.

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l can be calculated by keeping Fenwick tree in each slice, and the product $\cdot 2^{-(b-1)} \prod_{i=0}^{b-1} (\lceil \frac{n-j-1+i}{b} \rceil)!$ can be recalculating when we move j, because only one multiplier changes value.

The task is solved in $\mathcal{O}(n \log n)$ time.

B. Grid

In this problem we have a grid with a person looking in some direction in each cell of grid. We need to rotate persons minimal number of times so that there would be no two persons looking at each other.

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- with nm vertices in left part, corresponding to people in cells;
- with $4nm - (n-1)(m-1)$ vertices in right part, corresponding to all centers of each edge in grid;
- edges between vertices from left and right parts if the corresponding cell and edge are adjacent. The cost of edge should be equal to one of $\{0, 1, 2\}$ depending on the cost of turning a person in cell to look in this direction.

B. Grid

The maximal matching of minimal weight in this graph is exactly the optimal arrangement in the grid. There would be no two persons looking at each other, because each person will be directed somewhere, there would be no two persons looking at each other (no more than one man look at each side), and the cost of such arrangement is minimal.

Solution runs in $\mathcal{O}((nm)^2 \log nm)$ if we solve this problem by min cost max flow algorithm with Dijkstra.

C. Heap

In this problem we need to find a number of given d-ary heap in a set of all possible d-ary heaps.

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Suppose we have k heaps $h_1, h_2, \ldots, h_k$ with sizes $|h_1|, |h_2|, \ldots, |h_k|$ and the restrictions that in i-th heap all numbers should be at least x_i.

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Let $cnt_x^{(p)}$ be a number of elements with indices not less than p that are greater than x (and without the value v).

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Let's sort all heaps h_i such that x_i will be a non-increasing sequence. Than the number of ways to build the suffix of permutation is

$$\prod_{i=1}^{k} \binom{cnt_{x_i}^{(p)} - \sum_{j=1}^{i-1} |h_j|}{|h_i|} T(h_i)$$

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This formula follows from that we need to choose $|h_1|$ elements from the $cnt_{x_1}^{(p)}$ appropiate values for a set of values to first heap, and multiply it by $T(h_1)$. These $|h_1|$ elements can't be used in other heaps, and as $x_1 \geqslant x_i$ for $i \geqslant 1$, we won't be able to use these elements in all heaps, but they are counted in all $cnt_{x_i}^{(p)}$-s.

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The number of ways to construct a heap h_i with vertices $j = 1 \ldots |h_i|$ and sizes of subtrees $size_j$ is a

$$T(h_i) = \frac{|h_i|!}{\prod_{j=1}^{|h_i|} size_j}$$

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This solution works in $\mathcal{O}(n^3)$. To speed up it to $\mathcal{O}(n^2)$ we need to notice that the majority of binomial coefficients don't change between different values of v. The only thing that can happen is that some heap h_i will swap its order with subtrees of p, when v becomes less than x_i. Using this observation we can keep the current product of binomial coefficients and solve the problem in $\mathcal{O}(n^2)$ time.

D. Lines

We are given a n lines on plane, and we should select the maximum possible number of lines such that no two selected lines are parallel, and no two selected lines have an intersection at point with $x = 0$.

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Now, we can represent each line in form $y = kx + b$. Two parallel lines have the same k, and two lines with intersection at point $(0, b)$ have the same b. So, we should take the maximum number of lines with different k-s and b-s.

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We need to select maximum number of edges such that no two edges share a vertex. It is maximum matching problem in bipartite graph, and we can solve it with Kuhn's algorithm.

Solution with Kuhn's algorithm runs in $\mathcal{O}(nm)$ time.

E. Yet Another Problem About Permutations

In this problem we need to represent a given permutation as a product of minimal number of simple (each cycle has length no more than 2) permutations.

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Otherwise, answer is always 2. Let's show this in constructive way: solve the problem independently for all cycles. Suppose we have a cycle with length $l > 2$. We can draw all elements of this cycle on a circle in a vertices of regular polygon. We need to find a way to represent a cyclic shift of these vertices as a composition of two simple permutations.

We can make two symmetry operations with two lines as shown on the next slide.

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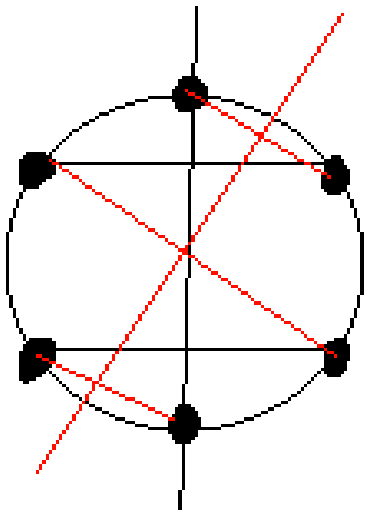

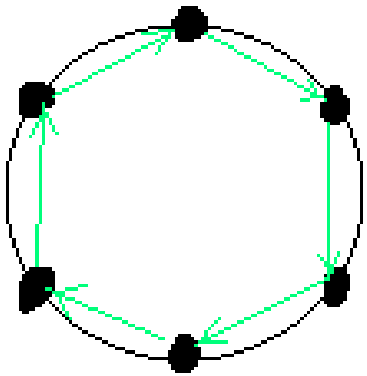

First, we make a symmetry with black line, and then symmetry with the red line. We can see, that both symmetries define a simple permutation (every element either remains on its place, or it is swapped with another element), and their composition (first black symmetry, then red symmetry) exactly shifts all vertices on cycle.

F. Strange Sequence

In this problem we need to find a length of n-th term in look-and-say sequence.

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Let $f(s)$ be a result of applying our operation to a string s, and $f(f(f \dots f(s)))$, where there are k iterations of f, be $f^{(k)}(s)$.

Let's look at the example: let $s = 13221133112$, and $f(s) = 11132221232112$. It can be proved that $f(s) = 1113222|12|32112$. In other words, $f^{(k)}(s) = f^{(k)}(13221133112) = f^{(k-1)}(11132221232112) = f^{(k-1)}(1113222) + f^{(k-1)}(12) + f^{(k-1)}(32112)$.

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We can see that the last symbol of $f^{(k)}(1113222)$ is always 2, and the first symbol of $f^{(k)}(12)$ is always 1 or 3.

F. Strange Sequence

So, we need to find a way to select a finite set of non-affecting strings $\{t_1, t_2, \ldots, t_n\}$ such that $f(t_i) = t_{j_1} t_{j_2} \ldots t_{j_{k_i}}$ for each i.

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It turns out that it is suffice to generate this set recursively: if we have some string t_i, we can just greedy divide $f(t_i)$ to substrings $a_1 a_2 \ldots a_t$ such that for each j the last symbol of a_j is always different from the first symbol of $f^{(k)}(a_{j+1})$ for each k. It is hard to check this condition for each k, but it is enough to check this condition for $k \leqslant 30$. We can add every string a_j from such partition to our set $\{t_i\}$ and call our partition function recursively for all new substrings.

F. Strange Sequence

After finish of the described algorithm, we get 93 different strings in the set $\{t_i\}$.

Now we can see that $|s_n|$ is a linear recurrence and we can compute n-th element of a recurrence in $\mathcal{O}(|\{t_i\}|^3 \log n)$ by binary exponentiation.

G. Shortest Accepted Word

In this problem we need to construct a lexicographically smallest word accepted by given regular expression.

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- from iteration vertex `*` we return `"$"`, because it is the lex.min word;
- from concatenation vertex `PQ` we return concatenation of their words;
- from union vertex `P|Q` we return the minimum of their words.

We can see that these rules construct the smallest words in each vertex. So, the only problem is to construct a parse tree, which can be done by recursive descent.

H. Work

In this problem we need to assign a set of works to workers, so the distribution is as closest is possible to uniform.

H. Work

Suppose i-th worker is assigned to x_i works. Then we need to minimize $\sqrt{\sum_{i=1}^{n}(x_i - \frac{m}{n})^2}$. It is equivalent to minimize its square: $\sum_{i=1}^{n}(x_i - \frac{m}{n})^2 = \sum_{i=1}^{n} x_i^2 - 2\frac{m}{n}\sum_{i=1}^{n} x_i + n\frac{m^2}{n}$. As each job that can be completed should be completed, we know that $\sum_{i=1}^{n} x_i = const$, so we need only to minimize $\sum_{i=1}^{n} x_i^2$ with given condition on the sum of x_i.

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- for each work j edge from its vertex to sink with $cap = 1, cost = 0$;
- for each worker i m edges from sink to its vertex; i-th edge with $cap = 1, cost = 2i - 1$.

H. Work

We can see that min cost max flow in this network is the solution of the task: each doable work will be done, and if worker i executes x_i works it will cost $\sum_{j=1}^{x_i} 2j - 1 = x_i^2$, so between all arrangements our algorithm would choose the one with minimal $\sum_{i=1}^{n} x_i^2$.

The solution works in $\mathcal{O}((n+m)nm)$, if we implement min cost max flow with Dijkstra's algorithm.