Research Project 9: Clique and Vertex Cover

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Chapter 1 ---- Background and Introduction

Background and Introduction:

Stephen Cook’s 1971 demonstration of the first (practically relevant) NP-complete problem, the Boolean Satisfiability problem make great differences on computational complexity. After this problem’s raise, the study on NP-complete started. The second year, 1972, Richard Karp took this idea a leap forward with his landmark paper, “Reducibility Among Combinatorial Problems”. And in this paper, he put forward 21 diverse combinatorial and graph theoretical problems:

1. Satisfiability
2. 0-1 integer programming
3. Clique
4. Set packing
5. Vertex Cover
6. Set Covering
7. Feedback node set
8. Feedback arc set
9. Directed Hamilton circuit
10. Undirected Hamilton circuit
11. Satisfiability with at most 3 literals per clause
12. Chromatic number
13. Clique cover
14. Exact cover
15. Hitting set
16. Steiner tree
17. 3-Dimensional matching
18. Knapsack
19. Job sequencing
20. Partition
21. Max cut

Then we will brief introduce these problems and provide the process of proving that the Clique problem is polynomially reducible to the Vertex Cover problem.
Chapter 2 ---- Brief Introduction of Karp’s 21 NP-complete problems

Brief Introduction of Karp’s 21 NP-complete problems:

1. **Satisfiability**: the Boolean satisfiability problem for formulas in conjunctive normal form (often referred to as SAT).

   **Description**: To justify is there an input make the given logical formula true?

2. **0-1 integer programming**: a branch of integer programming, we are asked to provide suitable value for some unknown variables.

   **Description**: To justify are there values consists of 0 and 1 can replace the unknown variables.

3. **Clique**: a complete subgraph in the given graph (find the largest subset, all of whose vertices are contiguous, in the graph).

   **Description**: Clique problem including finding the maximum clique, finding the maximum weight clique in a weighted graph, listing all maximal cliques, and solving the decision problem of testing whether a graph contains a clique larger than a given size.
4. **Set packing**: the set packing problem asks if some $k$ subsets provided as condition in the list are pairwise disjoint (pairwise disjoint: no pair of them have intersects).

**Description**: When comes to this NP-complete problem, you are given a list of sets and they are randomly, then you should choose sets as many as possible from them, which are pairwise disjoint.

5. **Vertex cover**: a set of vertices such that each edge of the graph is incident to at least one vertex of the set. And the minimum vertex cover problem can be formulated as a half-integral linear program whose dual linear program in the maximum matching problem.

**Description**: In this problem, you are asked to pick out least vertices, while for every edge in the given graph, we can find a vertex contiguous to it from the ones we picked out.
### Covering-Packing Dualities

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6. **Set covering**: You are asked to pick out some sets from the given group of sets, and they should contain all the elements of the group of sets.

**Description**: This problem is similar to Vertex cover. We can consider the sets as the edges contiguous to one vertex, and then our task is to solve the Vertex cover problem.

7. **Feedback vertex set**: A set of vertices containing at least one vertex from every cycle in the directed graph, and the minimum spanning tree (simply, if all the vertices in feedback vertex set are removed, the graph will become no circle).

**Description**: In this problem, our purpose is to broken all the circles in the graph, the method we can take is to delete the vertices in the graph. Besides, we should delete as less vertices as possible to reach the goal.
8. **Feedback arc set:** a set of edges which when removed from the graph, leave a DAG (directed acyclic graph).

**Description:** Make choices from the given graph’s edges, and the graph will become a DAG.

9. **Directed Hamilton circuit:** a special case of the traveling salesman problem, obtained by setting the distance between two cities to finite constant if they are adjacent and infinity otherwise.

**Description:** Find a path in the given graph, which can cover all the vertices. More detail description we can find from the pictures below.

10. **Undirected Hamilton circuit:** ditto.

**Description:** The same as the Directed Hamilton circuit except that we have no need to consider the edges’ direction. We can understand this problem more easily with pictures below.
11. **Satisfiability with at most 3 literals per clause:**

**Description:** We are required to find the possible assignments making the entire expression true. (Every clause contains 3 literals)

12. **Chromatic number** (also called the graph coloring problem): the smallest number of colors needed to color a graph G.

**Description:** Obviously, this problem is likely to the problem coloring the faces of a box or some other geometry without making two contiguous faces the same color.

13. **Clique cover:** determining whether the vertices of a graph can be partitioned into k cliques.

**Description:** Before solve this problem, we should clearly understand that clique is set of vertices, every vertices in which is contiguous to the others. And then we should try to divide the whole graph into several cliques, and the number of cliques is as less as possible when
the problem is optimization.

14. **Exact cover**: a decision problem to find an exact cover or else determine none exists.

**Description**: Select several sets from the sets list, which can cover all the elements appear among the sets.

Let $\mathcal{S} = \{A, B, C, D, E, F\}$ be a collection of subsets of a set $X = \{1, 2, 3, 4, 5, 6, 7\}$ such that:

- $A = \{1, 4, 7\}$;
- $B = \{1, 4\}$;
- $C = \{4, 5, 7\}$;
- $D = \{3, 5, 6\}$;
- $E = \{2, 3, 6, 7\}$; and
- $F = \{2, 7\}$.

Then the subcollection $\mathcal{S}^* = \{B, D, F\}$ is an exact cover, since each element in $X$ is contained in exactly one of the subsets:

- $B = \{1, 4\}$;
- $D = \{3, 5, 6\}$; or
- $F = \{2, 7\}$.

15. **Hitting set**: given a collection of sets, a set which intersects all sets in the collection in at least one element is called a hitting set.

**Description**: Pick out as less elements as possible from all sets to construct a new set which contains at least one shared element with all the sets.
16. **Steiner tree**: given a set $V$ of points, interconnect them by a network of shortest length, where the length is the sum of the lengths of all edges.

**Description**: From the pictures below we understand Steiner tree easier.

![Steiner tree diagram](image)

17. **3-dimensional matching**:

**Description**: Find as many connections as possible between three sets of points.

![3-dimensional matching diagrams](image)
18. **Knapsack**: Given a set of items, each with a weight and a value, determine the number of each item to include in a collection so that the total weight is less than a given limit and the total value is as large as possible. It derives its name from the problem faced by someone who is constrained by a fixed-size knapsack and must fill it with the most useful items.

19. **Job sequencing**:

   **Description**: Provide a method to list the tasks, and complete the tasks in the sequence. This problem is seriously considered in the processes management.

20. **Partition problem**: whether a given multiset of integers can be partitioned into two "halves" that have the same sum.

   **Description**: Firstly, you will be provided a collection of sets of integers, you should to try to divide these sets into two parts, which have the equal sum.
21. **Max cut**: a maximum cut is a cut whose size is not smaller than the size of any other cut. The problem of finding a maximum cut in a graph is known as the max-cut problem.

**Description:**

![Diagram of a graph with a maximum cut highlighted in red.]
Chapter 3 ---- Clique and Vertex Cover

Clique and Vertex Cover:

Clique

a complete subgraph in the given graph

The brute force algorithm finds a 4-clique in this 7-vertex graph (the complement of the 7-vertex path graph) by systematically checking all \( C(7,4)=35 \) 4-vertex subgraphs for completeness.
Vertex Cover

Formally, a vertex cover of a graph $G$ is a set of vertices $C$ such that each edge of $G$ is incident to at least one vertex in $C$. The set $C$ is said to cover the edges of $G$. The following figure shows examples of vertex covers in two graphs (the set $C$ is marked with red).

A minimum vertex cover is a vertex cover of smallest possible size. The vertex cover number $\tau$ is the size of a minimum vertex cover. The following figure shows examples of minimum vertex covers in two graphs.

Polynomial Reducibility from Clique problem to Vertex Cover problem

The complement $G^c$ of a graph $G$ contains exactly those edges not in $G$.

Conclusion: $G$ has a clique of size $k$ iff $G^c$ has a vertex cover of size $|V| - k$.
• Claim: If G has a clique of size k, $G_C$ has a vertex cover of size $|V| - k$

  ■ Let $V'$ be the $k$-clique

  ■ Then $V - V'$ is a vertex cover in $G_C$
    ○ Let $(u,v)$ be any edge in $G_C$
    ○ Then $u$ and $v$ cannot both be in $V'$
    ○ Thus at least one of $u$ or $v$ is in $V' - V'$, so
      edge $(u, v)$ is covered by $V - V'$
    ○ Since true for any edge in $G_C$, $V - V'$ is a vertex cover

• Claim: If $G_C$ has a vertex cover $V' \subseteq V$, with $|V'| = |V| - k$, then G has a clique of size $k$

  ■ For all $u,v \in V$, if $(u,v) \in G_C$ then $u \in V'$ or $v \in V'$ or both

  ■ Contrapositive: if $u \notin V'$ and $v \notin V'$, then $(u,v) \in E$

  ■ In other words, all vertices in $V - V'$ are connected by an edge, thus $V - V'$ is a clique

  ■ Since $|V| - |V'| = k$, the size of the clique is $k$
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Declaration:

*We hereby declare that all the work done in this project titled “Clique and Vertex Cover” is of our independent effort as a group.*

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